# Inversion in a Circle

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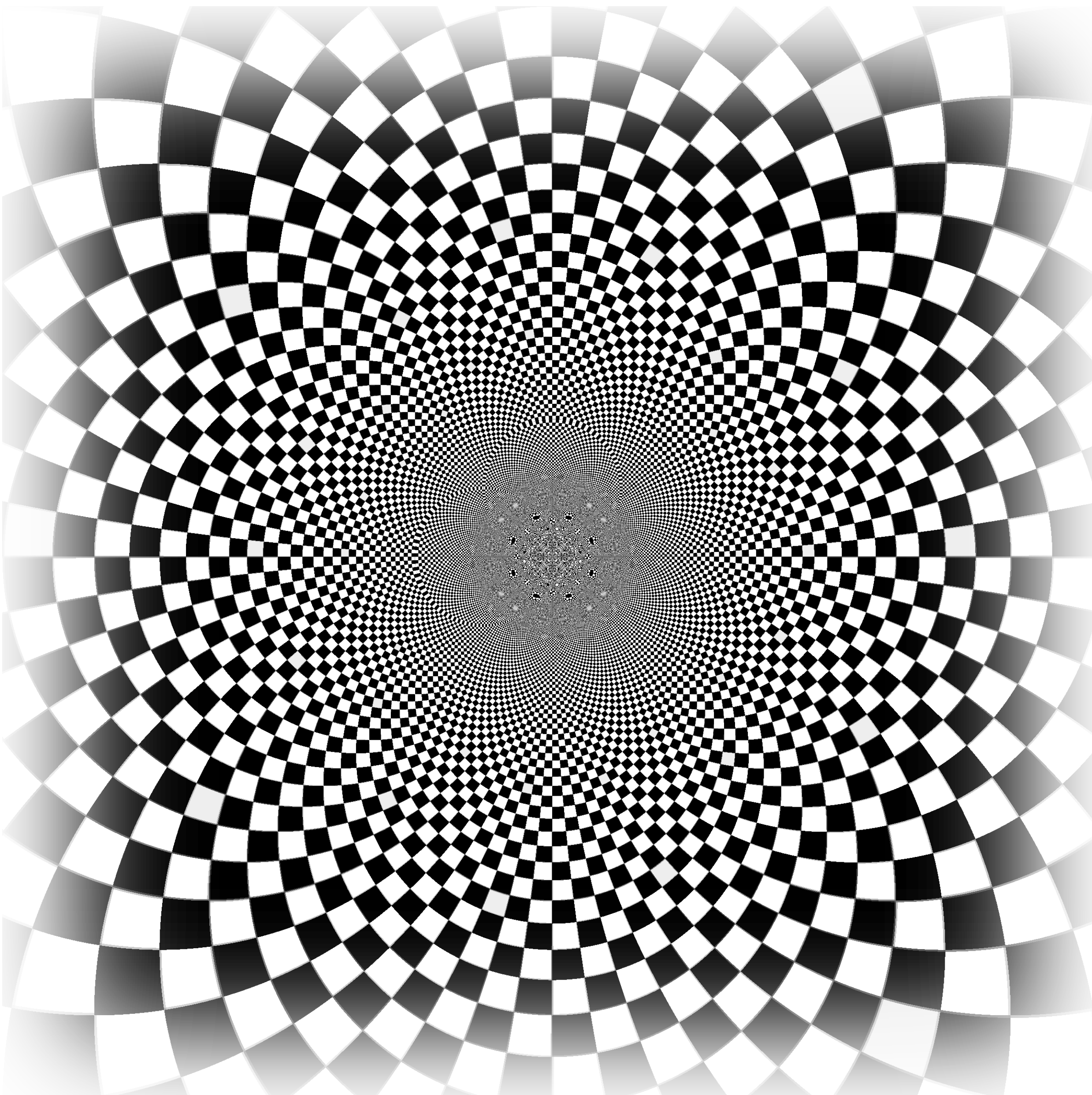
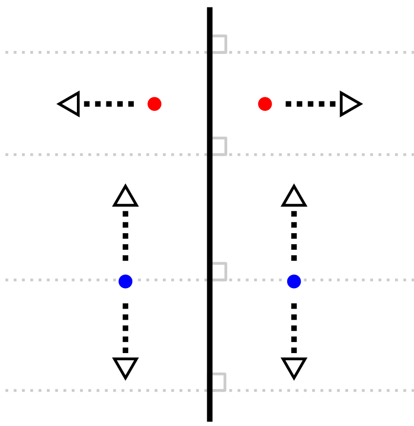


## Introduction

Inversion in a circle is a transformation process that "inverts" the plane by swapping points which are inside a chosen circle (called the circle of inversion) with points which are outside that circle. The process is quite simple, but the consequences for things such as lines, circles, angles, and even inversion itself are both entertaining and enlightening[1].

## The Mirror Analogy

To get an intuitive understanding of inversion in a circle, let's begin with a kind of inversion that is familiar to many people: mirroring of the plane about a line.

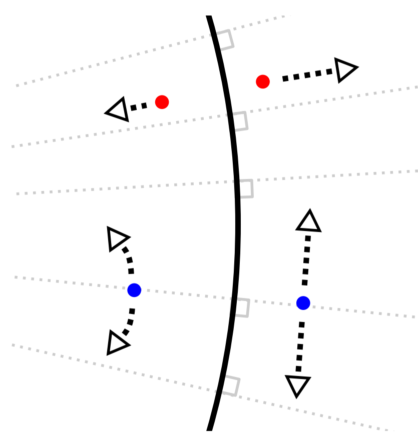


Imagine that we've divided the plane into two halves using a line, called the line of inversion (solid). The left-hand side of the plane is a mirror image of the right-hand side, with the properties we'd expect of a typical mirror.

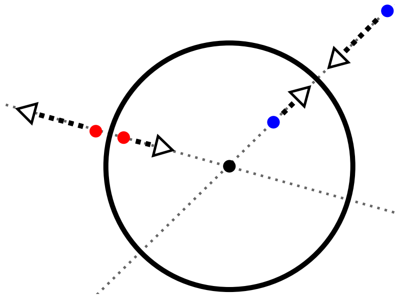
For example, let's pick two points (`red` and ***blue***), move them around, and watch what their inverted twins do. The ***blue point's*** motion is parallel to the line of inversion (or mirror), and its twin follows it along the mirror exactly. The `red point's` motion is perpendicular to mirror and, as expected, both it and its twin get closer to the mirror or farther away from it at the same rate.

What if we were to bend our mirror slightly? For starters, the dotted lines that were perpendicular to the straight-line mirror are now perpendicular to the tangent lines at their intersections with the curved mirror. These lines also meet at a point finitely far away now.

In fact, our bent mirror looks and acts a great deal like a portion of a very large circle whose center is due west!



[1]The picture on the title page was created via computer by tiling the plane with a checkerboard pattern and then inverting it in a circle centered on the origin. Can you guess where the boundary of that circle lies?

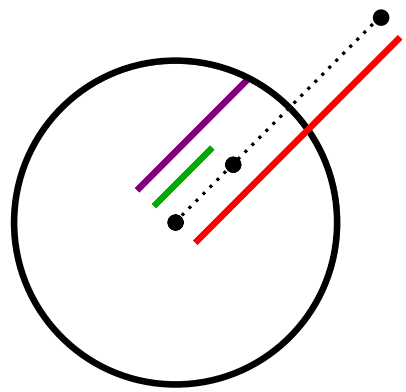


Let's zoom out and look at our mirror as a completed circle. How has the motion of our points changed?

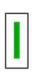
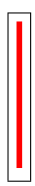
Motion towards the mirror (***blue points***) and away from the mirror (`red points`) follows lines that pass through the circle's center. In addition, a point and its inverted twin move at different rates, as the "focal point" of our mirror is now a finite distance from its edge. In the next section, we'll delve into the geometric reasons for these phenomena.

## Getting Geometric

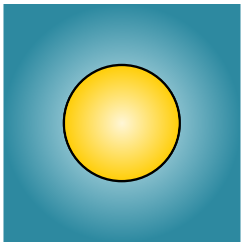
As a first step into more formal territory, let's examine the details behind inversion of a single point in a circle. With our original straight-line mirror, a point and its twin were always the same distance from the mirror. With a circle-mirror, things have changed: the center of the circle is analogous to the point at infinity, meaning the distances a point and its twin are from the mirror's edge cannot always be equal.



Specifically, the **distance from the circle's center to the inside point** and the **circle's radius** have the same ratio as the **radius** and the **distance from the center to the outer point**.

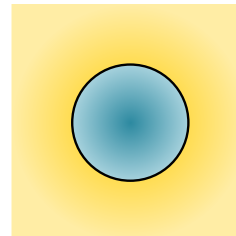
In other words,  is to  as  is to . This means that as points outside the circle get farther and farther away, their inverted twins get closer to the circle's center at a slower and slower rate. It can be difficult to visualize, though, when the plane and circle of inversion are both painted white.

To remedy this, let's imagine we could see the entire plane in a single picture. We'll paint both our circle of inversion and the plane with different colors, fading them according to distance from the origin. Then, we'll invert the plane in that circle and observe how the colors get rearranged (see next page).

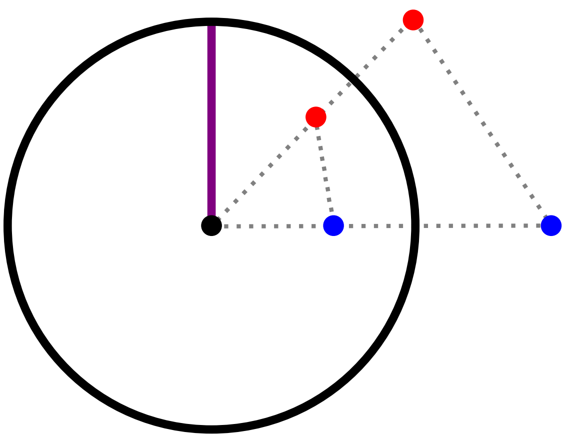


On the $\overleftarrow{\text{left}}$, we can see the plane colored with blue and the circle of inversion colored with yellow. Notice that both colors get darker as they get further from the origin (center).

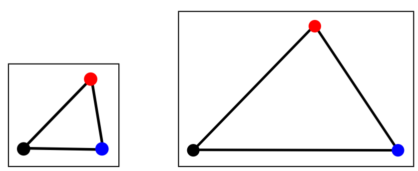
On the $\overrightarrow{\text{right}}$ is the inverted plane in the circle. The inside and outside of the circle have been exchanged. The dark blue points, previously at the old plane's edges, are now at center of the circle, and the golden yellow points at the old circle's boundary now form a halo around it.



## Similar Triangles



Consider the ***blue*** and `red` points outside this circle and their inverted twins inside. By adding in the dashed lines, two triangles emerge, each consisting of the **center point**, a `red` point, and a ***blue*** point:



Because of the link between inverted twins, we know that the following relationships hold:

$$\frac{\textbf{center}\ \text{to}\ \texttt{inside}}{\textbf{radius}} = \frac{\textbf{radius}}{\textbf{center}\ \text{to}\ \texttt{outside}} \qquad \frac{\textbf{center}\ \text{to}\ \textbf{\textit{inside}}}{\textbf{radius}} = \frac{\textbf{radius}}{\textbf{center}\ \text{to}\ \textbf{\textit{outside}}}$$

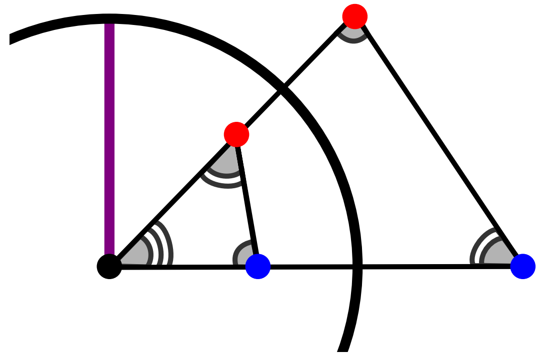
A tiny bit of manipulation yields something interesting:

$$\frac{\textbf{center}\ \text{to}\ \texttt{inside}}{\textbf{center}\ \text{to}\ \textbf{\textit{outside}}} = \frac{\textbf{center}\ \text{to}\ \textbf{\textit{inside}}}{\textbf{center}\ \text{to}\ \texttt{inside}}$$

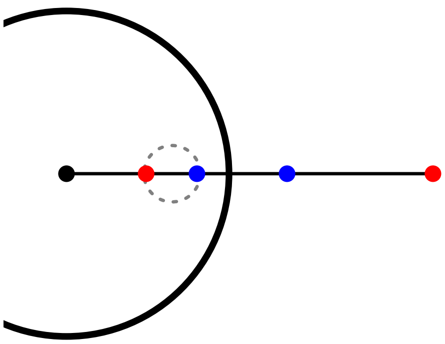
From here, it's easy to see that the **center** to `inside` side of the small triangle corresponds to the **center** to ***outside*** side of the large triangle, and the **center** to ***inside*** side of the small triangle corresponds to the **center** to `outside` side of the large triangle.

Since both triangles share the angle at the **center**, we can conclude that they are similar by side-angle-side.

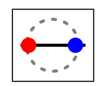
Putting it all together, the angle correspondence on the right holds for any two pairs of inverted twins. We can use this result immediately to show that circles always invert to circles.



## Preservation of Circles



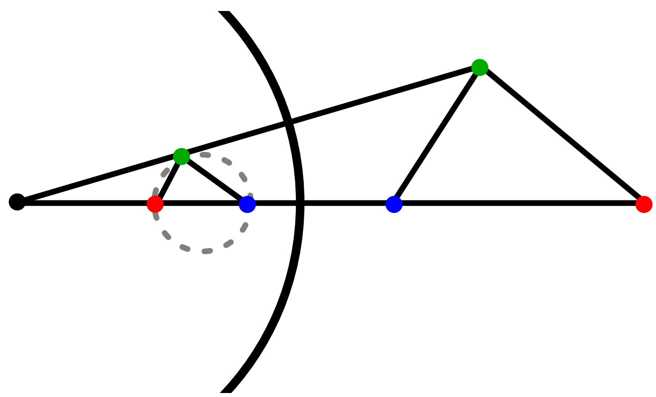
On the left, we have our circle of inversion (solid) and two pairs of inverted twins (`red` and ***blue***). Inside the circle of inversion, there is a smaller circle (dashed) whose diameter is given by the inside points. That is,

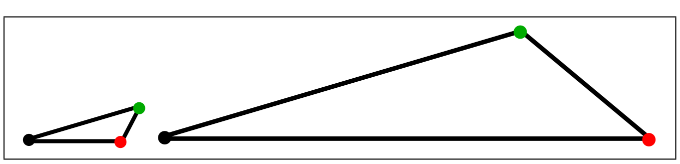
`inside` to ***inside*** is a diameter of  .

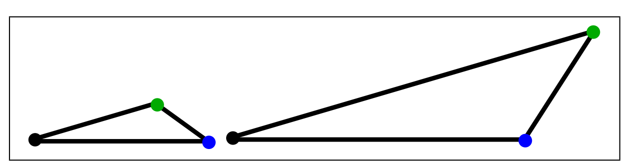
Let's pick a **random point** on the inner circle, construct its inverted twin, and connect our points with lines as shown ↓

This construction should look familiar from the previous section with the addition of two triangles due to the **random point**.

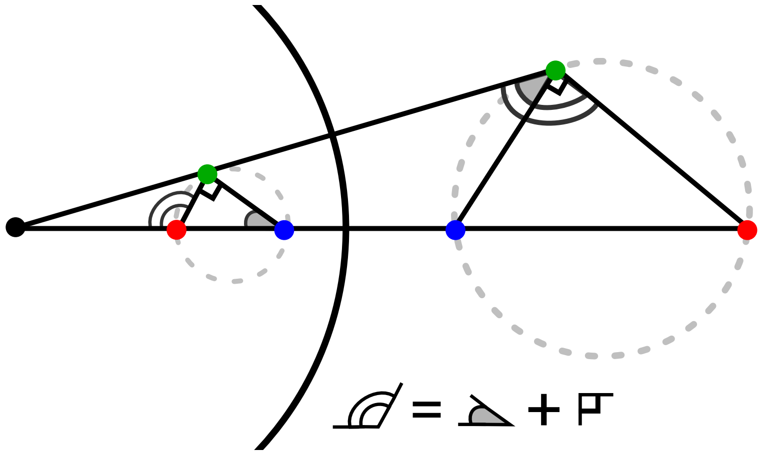
In total, we have four triangles, grouped below as two pairs of similar triangles.







After filling in our similar triangle angles, we can make quick work of this problem by using the fact that `inside` to ***inside*** is a diameter of the inner circle.



The double-lined angle is exterior to the triangle inscribed on the inner circle, making it the sum of the shaded angle and 90°.

Thus, we can carry the 90° angle outside and prove that our **random point's** twin lies on a circle which has ***outside*** to `outside` as a diameter.

Remember, we choose our **point** at random, so this means any point on the inside circle will lie on the outside circle along a ray cast from the **center** of the circle of inversion.

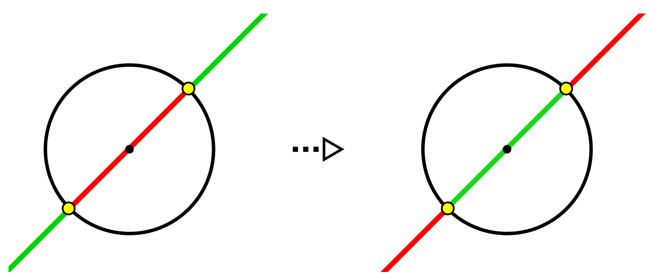
## Lines are Circles

From the handful of facts we have learned so far, it is possible to infer the behavior of inversion in a circle for several special cases. One conceptual hurdle that must be cleared, however, is the idea that *lines are circles whose centers are infinitely far away.* From a local perspective, the circle appears as a straight line because any indication of the curve would contradict the center being infinitely far away.

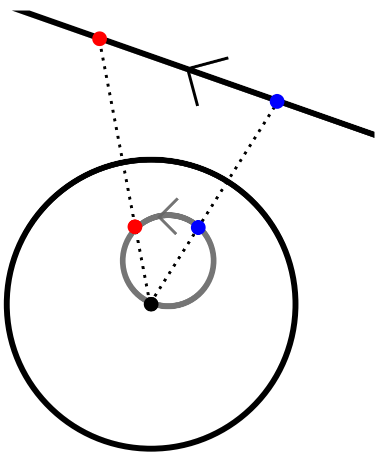
With this in mind, what would we expect lines which pass through the center of our circle of inversion to invert to? If lines are infinite circles, circles invert to circles, and our line touches the center (whose inverted twin is infinitely far away), then our line's inversion is the same line![2]

This "same" line, however, has its points **outside the circle** swapped with its points **inside the circle** as demonstrated on the right.

The points that lie on the circle of inversion itself, of course, remain fixed, as they are their own inverted twins.



[2]Another way to think of it is that a point and its inverted twin must be co-linear with the center. If a line passes through the center, it's not possible to stray from that line during inversion.



After discussing lines which pass through the **center** of the circle of inversion, it's natural to wonder about lines which do not.

Once again, we already know that lines are circles and circles invert to circles. Our original line extends to infinity, so its inversion must touch the **center**. Combined with the knowledge that a point and its inverted twin are co-linear with the **center**, we have everything needed to construct the diagram on the left.

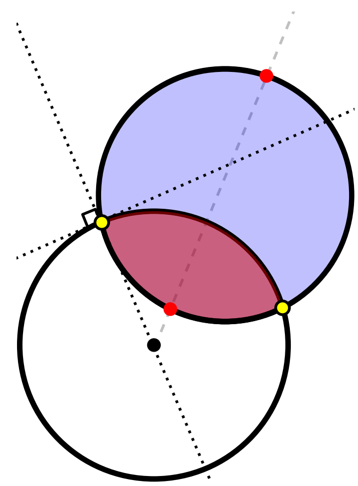
Thus, lines that do not pass through the **center** are inverted to circles that do.

## Orthogonal Circles

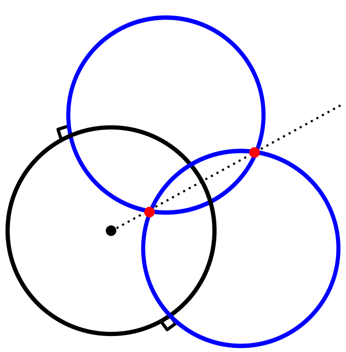
A particularly noteworthy case of the preservation of circles during inversion is when the circle we're inverting is orthogonal to the circle of inversion (i.e. the tangent at their intersection).

Orthogonal circles invert to themselves, and this can be proven by remembering three facts about inversion in a circle:

1. Circles invert to circles
2. Points on the circle of inversion invert to themselves
3. Pairs of inverted twins are co-linear with the **center**



These three facts serve to constrain an orthogonal circle's inverse to be a single circle: itself. Note, of course, that the interior of this circle is exchanged about the circle of inversion.



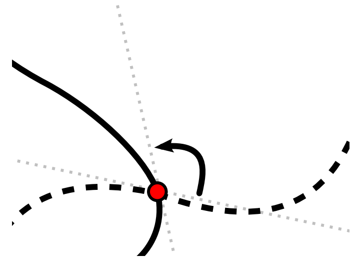
We can actually construct a `point's` inverted twin using just orthogonal circles! After drawing an ***orthogonal circle*** through our `point`, we need only find a second ***orthogonal circle*** that also passes through it.

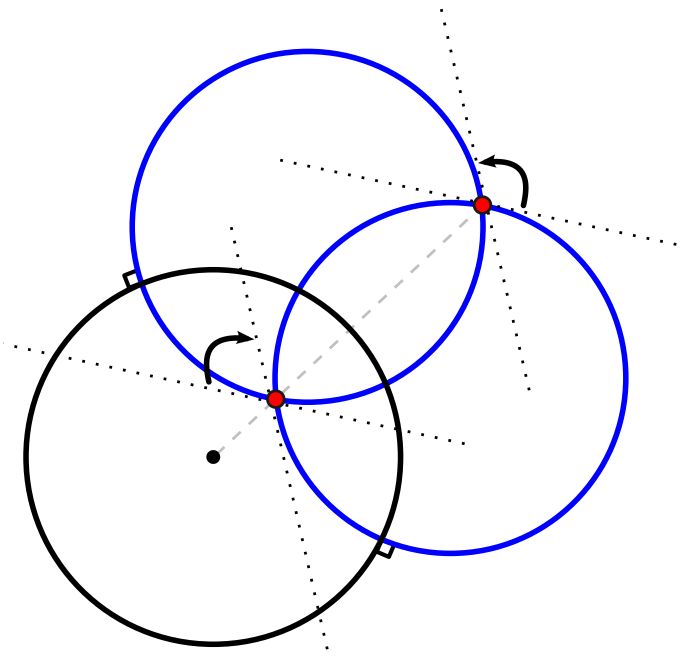
Our `point's` twin will lie at the other intersection of our two ***orthogonal circles***. This is an immediate consequence of the fact that orthogonal circles invert to themselves and that incidence is preserved during inversion.

## Preservation of Angles

Inversion in a circle preserves angles between curves, such as the two pictured to the right (solid and dashed).

Precisely, this is the angle between the tangent lines (dotted) at the point of intersection.





We can show angle preservation by replacing our curves with two ***orthogonal circles*** that are tangent to our tangents and intersect at the same point.

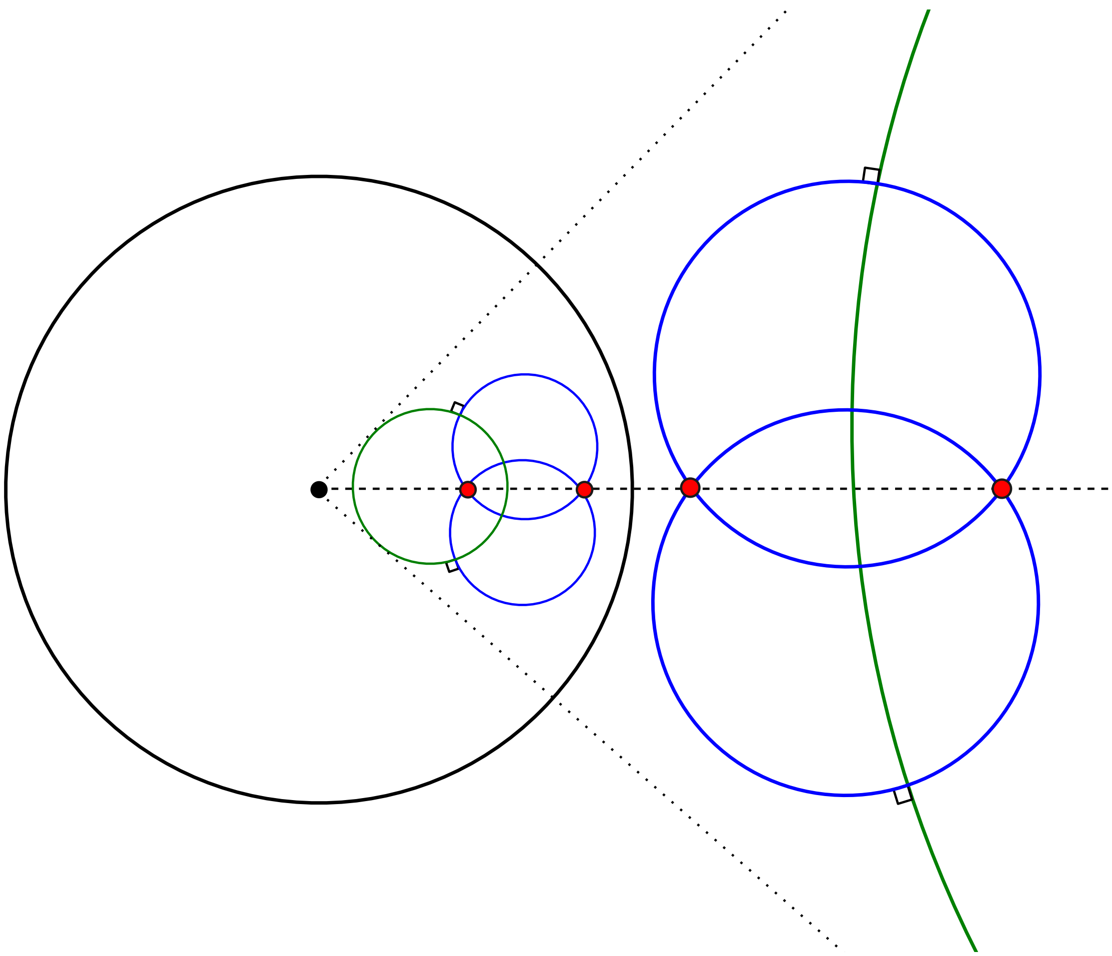
As in the previous section, we know that the other point where our ***orthogonal circles*** intersect is the inversive twin. With the aid of the diagram, it's trivial to see that the tangency relationship will be preserved by symmetry and thus so will our angle (albeit, going the other way).

As there are an infinite number of orthogonal circles relative to a circle of inversion, this technique will work for any two curves.

## Preservation of Inversion

For the capstone of this paper, we will now demonstrate that inversion itself is preserved during inversion. This is an especially beautiful result, and its proof is as simple as it is elegant.

To start, we need only remind ourselves that points on the circle of inversion invert to themselves, as do orthogonal circles. With this in mind, we can imagine inverting a trio of intersecting circles as shown on the next page.



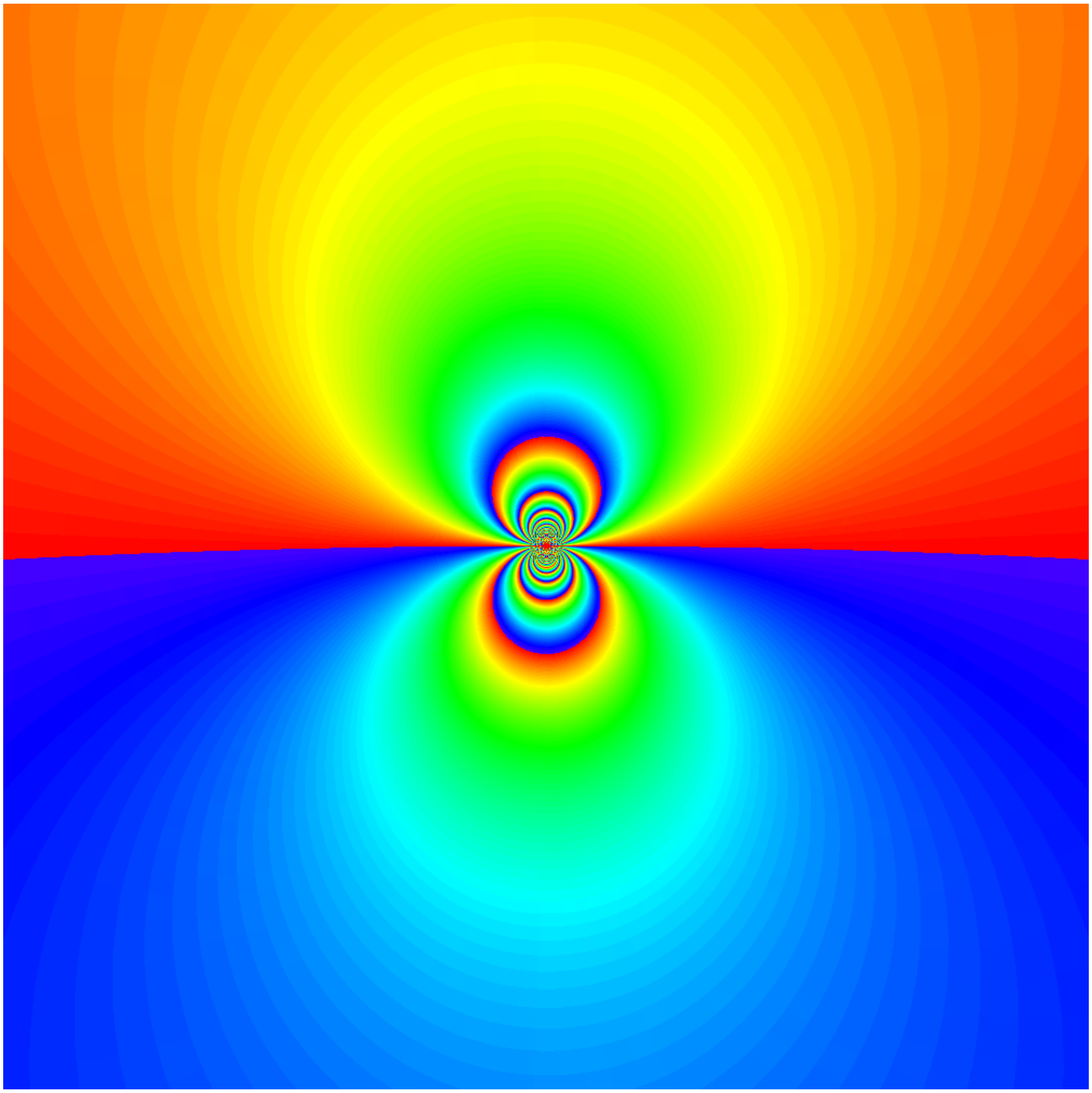
Here we see our trio of circles (one **small green**, two ***small blue***) inverted about the black circle. Notice that the two ***small blue circles*** are orthogonal to the **small green circle**, making the two `red points` on the left inversive twins in the **small green circle**.

Knowing that circles invert to circles, and that both incidence and angles are preserved is enough to prove that the inverted trio maintains the orthogonal relationship between the two ***large blue circles*** and the **large green circle**. As such, if we now think of the **large green circle** as the circle of inversion, it is easy to see that the two `red points` on the right are also inversive twins.

With this fact, we are free to apply inversion in different circles in any order we please, knowing that the inversive relationship between points will be preserved throughout.

In preparing for this paper, I spent time writing a program that would invert images in a circle for me so that I could experience it firsthand. The cover of this paper and a few of the images throughout were generated using it.

For my final page, I'd like to share one image that was generated by tiling the plane with rainbow-colored squares and inverted in a small circle at the center. I found it to be very pleasant, and when rotated on its side (as below), it reminded me of the sun reflecting off the ocean. Perhaps I'll title it **Rainbow Ocean**.



*Thank you,*
*Michael Hansen*